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THEORY OF FLOATING TUBES

By Frank Gilman

Article 1. It is proposed to treat of the theory of floating tubes or rods as applied to the measurement of the velocity of water in open channels; but, before doing so, we shall give a brief historical sketch of the use of this method in water measurements.

The first public mention of the method, so far as known, was made by T. A. Mann, a member of the Imperial and Royal Academy of Sciences at Brussels. He communicated a paper to the President of the Royal Society of London, Joseph Banks, who read it before the Society on June 24, 1779. The subject of the paper was "The Hydraulics of Rivers and Canals," and in it a method is described for measuring the velocity of water in canals by means of wooden rods, or poles, of a length somewhat less than the depth of the water, while at the lower end are suspended as many small weights as may be necessary to keep the rod in a vertical position. He advises that a small straight wire be fastened to the center of that end of the rod which projects from the water so that it may indicate the deviations of the rod from a vertical position, and enable inferences to be made in regard to the relative velocities of the water at different depths.

Mr. Mann speaks of this method as the best and simplest of which he knows for measuring the velocity of the water in canals and rivers. He refers to it as if it were a well known method, but gives no example of its application. It is probable, however, that he had used it, for he says that he had long lived in a country that abounded in canals, and had been much employed in matters relating to hydraulics.

The results of the application of this method were first published by C. R. T. Krayenhoff in Amsterdam, Holland, in 1813. His work gives an account of his observations on the hydrography and topography of Holland. The floats that he used were wooden poles loaded with lead at the bottom, and carrying copper floats at the surface. The next application of the method was made by M. de Buffon, who in 1821 gauged the Tiber by the use of bundles of rods loaded at the lower ends and extending from the surface nearly to the bottom.

In 1835 Destrem gauged the Neva by the same method.

In measuring the discharge of small canals Hirn used light covered frames, so arranged as to be at right angles to the current, and of such extent that they nearly filled the whole cross-section of the stream, and consequently gave, at one reading, the approximate mean velocity at that section.

In 1852 James B. Francis, of Lowell, Mass., made most elaborate experiments on measuring the discharge of canals by the use of loaded tubes. These experiments were supplemented by others made in 1856, and a full account of all of them can be found in "The Lowell Hydraulic Experiments," a second edition of which was published by Van Nostrand in 1868. Mr. Francis compared the discharge given by the use of tubes with the same quantity passing over a weir and determined by weir measurements, and found that the difference was generally less than two per cent.

The tubes used were hollow tin cylinders, two inches in diameter, soldered together, with a solid cylindrical piece of lead, of the same diameter, at the lower end. The tubes ranged in length from 6 to 10 feet, and the centers of gravity of the longer tubes were about 1.9 feet from the lower ends. Two beams were laid across the canal, at right angles to the current and 70 feet apart. These constituted the upper and lower transit stations. The time occupied by the tube in passing from one station to the other, was determined by means of a stop watch, or chronometer; and from this and the known distance the velocity of the tube was calculated. The up-stream side of each beam was figured from one end to the other, so that the distance from the left shore (looking down stream) at which the tube passed each beam could be noted. The results were plotted on cross-section paper, calling the mean distance from the left shore the abscissa, and the velocity the ordinate. Between the points thus obtained a curve was drawn so that the sum of the vertical distances from the curve of the plotted points which are above, should be equal to the sum of the vertical distances of the points which are below the curve, or so that the sum of the positive errors should be equal to the sum of the negative errors.

From the mean velocity curve, thus obtained, readings were taken at intervals corresponding to one foot each in the width of the stream. The sum of these readings multiplied by the mean depth gave the discharge.

Very elaborate experiments in measuring the discharge of streams by the use of rods were made by Capt. Allan Cunningham near Roorkee, India, on the Ganges Canal. These experiments were begun in 1874 and continued, with some intermissions, until 1879. Capt. Cunningham found that rods moved

more steadily than any other sort of float, that they gave the result more rapidly, were more easily handled and less delicate, being simple in construction. He recommends that for measurements of mean velocity past a vertical, the rods should supersede all other instruments in cases favorable to their use. The conditions favorable to their use are that the cross-section and declivity of the stream should be uniform for a considerable distance, that the bottom should be free from obstructions, and that the depth should not exceed 15 feet.

M. A. Graëff, in his "Traité d'hydraulique," published in 1883, expresses a similar opinion in regard to the merits of the loaded tube, or rod. He states that Italian engineers had adopted the method of measuring the mean velocity by means of rods, and that he himself had used it in gauging the discharge of the Loire and its tributaries.

The Mississippi River experiments of Messrs. Humphreys and Abbot remain to be mentioned, which are the most important of all, as far as this paper is concerned, since by means of them the truth of our fundamental formula, $v = a + bx + cx^2$, was first demonstrated, in which v denotes the velocity of the water at the depth x, the total depth being unity, while a, b, and c are constants determined by experiment. The experiments were made át Carrolton and Baton-Rouge, Louisiana, in 1851. The mean depth of the river was about 82 feet. This depth was divided into 10 equal parts, and at each proportional depth 222 observations of the velocity were made, and their mean taken as the true velocity.

The results are tabulated on page 244 of the "Report of the Mississippi River," published at Washington in 1876. The mean results are given further on in this article, where it is seen that they satisfy the parabola equation with considerable exactness. Long tubes, of course, could not be used in measuring the velocity at different depths, nor in gauging the discharge for a river of such depth as the Mississippi. The apparatus used was a double float, consisting of a surface-float, a sub-float, and a connecting cord. The surface-float was of cork, 5 inches long, 1 inch thick, and submerged to a depth of $1\frac{1}{2}$ inches. Its weight, therefore, was not more than one-fourth of a pound.

A wire one foot in length, and carrying a small flag, was inserted in this float. The sub-float was a keg, open at both ends, beveled at the lower edge, and weighted with strips of lead to keep it in a vertical position. The weight of the keg, with the ballast, was about 9 pounds. Its diameter was 10 inches and its height 15 inches, thickness of staves three-eighths of an inch. The connecting cord was of hemp and one-tenth of an inch in diameter.

Its weight when stretched to its full length of 90 feet, was one-half a pound. These experiments have been severely criticized, on the ground that the sub-velocity measurements, especially those at great depths, did not truly represent the velocities at these points, on account of the disturbing influences due to the surface float and connecting cord. But these criticisms do not seem to be well founded, as a little calculation will show that the vis viva of the surface-float and cord due to the difference between the velocity of the water at the bottom, and the mean velocity of the water surrounding the cord, would be less than one-tenth of one per cent. of the total vis viva of the keg. Another result of the Mississippi River experiments was the conclusion that there is a nearly constant ratio between the mid-depth velocity and the mean velocity past a vertical line from the surface to the bottom. This follows however, from the form of the equation which gives the relation between the velocities at different depths, viz., $v = a + bx + cx^2$. Calling v_m the mean velocity past a vertical line, and $v_{\frac{1}{2}}$ the mid-depth velocity, we evidently have

$$\frac{v_m}{v_{\frac{1}{2}}} = \frac{a + \frac{b}{2} + \frac{c}{3}}{a + \frac{b}{2} + \frac{c}{4}}\,;$$

substituting the values of a, b, and c as deduced from the Mississippi River experiments, viz:

$$a = +\,3.1952, \qquad b = +\,0.4424, \qquad c = -\,0.7652,$$

we find $v_m/v_{\frac{1}{2}} = 0.980$.

If this ratio, 0.98, be applied to Capt. Cunningham's first 46 series of experiments, and each of the mid-depth velocities be multiplied by 0.98, the results will differ from the v_m of his experiments by less than one per cent in the majority of cases. The following table gives a synopsis of the results of the Mississippi River experiments:

Relative Depth	Observed velocity in feet per sec.	Velocity by formula
x	v	$v = a + bx + cx^2.$
0	3.1950	3.1952
.1	3.2299	3.2318
.2	3.2532	3.2531
.3	3.2611	3.2591
.4	3.2516	3.2497
.5	3.2282	3.2251
.6	3.1807	3.1852
.7	3.1266	3.1299
.8	3.0594	3.0594
.9	2.9759	2.9735

ARTICLE 2. In discussing the theory of floating tubes different results will be obtained according to the law assumed to hold true for the resistance of fluids. Some engineers assume that the resistance of fluids varies as the first power of the velocity, and consequently they take it for granted that the velocity of a floating tube is the same as the mean velocity of the water in the same vertical line of the same length as the tube. It would follow from this that the proper length of a tube, theoretically, for measuring the velocity of a current, is the same as the depth. But since in practice the length of the tube must be less than the depth, it is inferred that the velocity of the tube is always greater than the mean velocity of the current, in consequence of the slower motion of the water along the bottom, and tables of corrections are prepared to be applied to the observed velocity of the tube, which corrections are always negative. It is easily shown by analysis that this assumption and practice would be correct if the impulse and resistance of fluids varied directly as the velocity. But according to the law accepted by the great majority of experimenters, the impulse and resistance vary as the square of the velocity; and not only has this been demonstrated by experiment, but it has been shown to be true from theoretical considerations.*

* For a theoretical proof, see Weisbach's *Mechanics*, Vol. 1, Article 498.

We will now investigate the conditions for determining the velocity of a tube when immersed vertically in a stream of water, on the hypothesis that the resistance generated by the motion of a solid body in a still fluid is proportional to the square of the velocity, and that when the fluid itself has motion the resistance is proportional to the square of the relative velocity of the solid and fluid.

In the case of a vertical tube borne along by the current some parts of the tube will move faster than the adjacent fluid, and some slower. The parts that move faster will meet a force of resistance, and those that move slower a force of acceleration, each of which will be proportional to the square of the relative velocity of that portion of the tube and adjacent fluid.

We shall discuss in this article the case in which the velocity of the tube is less than the velocity of the water at the surface, as represented in figures 1 and 2, in which LP denotes the tube, AB the surface of the water, ACI the parabolic curve of velocity, EF the velocity v, corresponding to the depth BF as x, the relation of v and x being expressed by the formula, $v = a + bx + cx^2$.

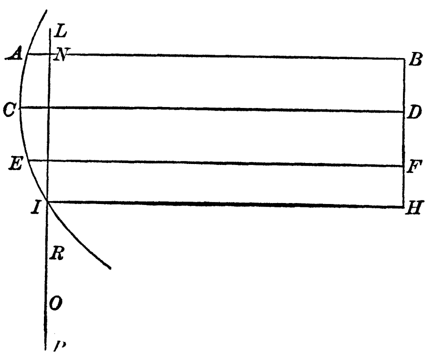

FIG. 1.

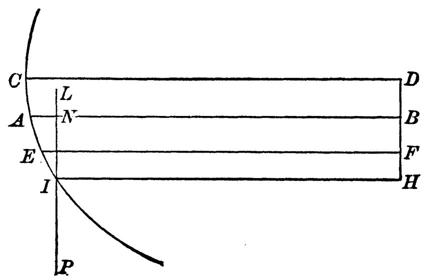

FIG. 2.

Figure 1 represents the case in which the axis of the parabola, CD, (which is the maximum velocity line) is below the surface of the water, while figure 2 represents the case when this axis is above the surface.

In the former case b/c is negative, in the latter positive. When the tube has attained a state of uniform motion, the sum of the forces of resistance will be equal and opposite to the sum of the forces of acceleration, a condition expressed by the following equation :

$$\int_0^h (v - v')^2 dx = \int_h^l (v' - v)^2 dx,$$

in which h denotes the distance from the surface to the point where the velocity of the water is equal to the velocity of the tube, and is represented in the figures by BH; v' denotes the velocity of the tube, and is equal to $a + bh + ch^2$, while l is the length of the submerged portion of the tube, taken on the same scale as x, the total depth being unity.

Substituting the values of v and v', as above given, performing the indicated operations, and arranging the results with reference to the powers of h, we have

$$\text{(1)} \quad \begin{aligned} &\frac{16}{15} h^5 + \left(\frac{5}{3}\frac{b}{c} - l\right) h^4 + \left(\frac{2}{3}\frac{b^2}{c^2} - 2\frac{b}{c} l\right) h^3 + \left(\frac{b}{c} l^2 + \frac{2}{3} l^3 - \frac{b^2}{c^2} l\right) h^2 \\ &\qquad + \left(\frac{b^2}{c^2} l^2 + \frac{2}{3}\frac{b}{c} l^3\right) h = \frac{1}{3}\frac{b^2}{c^2} l^3 + \frac{b}{2c} l^4 + \frac{l^5}{5}. \end{aligned}$$

Let $h = rl$, and substitute this value of h in the above equation, then, dividing through by l^5, and representing b/cl by s, we obtain the following for the final equation:

$$\text{(2)} \quad \begin{aligned} &\frac{16}{15} r^5 + \left(\frac{5}{3} s - 1\right) r^4 + \left(\frac{2}{3} s^2 - 2s\right) r^3 + \left(s - s^2 + \frac{2}{3}\right) r^2 \\ &\qquad + \left(\frac{2}{3} s + s^2\right) r - \frac{s^2}{3} - \frac{s}{2} - \frac{1}{5} = 0. \end{aligned}$$

It is easily shown, by Sturm's Theorem, that this equation will give only one real value for r corresponding to any real value of s; for instance let $s = -1$, then the equation becomes

$$r^5 - 2.5r^4 + 2.5r^3 + 0.625r^2 + 0.3125r - 0.03125 = 0.$$

Call this equation X, and its first derived polynomial X_1. Then finding the greatest common divisor of X and X_1, denoting the successive remainders, with their signs changed, by R, R_1, R_2, etc., and writing the results in two rows, we have

X	X_1	R	R_1	R_2
r^5	r^4	$-r^2$	$-r$	$+1.55164$

Each term in the second row denotes the first term of the equation designated by the symbol immediately above.

Substituting successively $-\infty$ and $+\infty$ for r in the equations X, X_1, R, R_1, and R_2, we have the following signs:

for $r = -\infty$	$- + - + +$	three variations;
for $r = +\infty$	$+ + - - +$	two variations;

the equation has therefore one real root.

Next substitute $+1$ for s in equation (2), and by a similar process we obtain the following results:

	X	X_1	R	R_1	R_2	R_3	
	r^5	r^4	r^3	$-r^2$	r	$+ 0.58$	
for $r = -\infty$	$-$	$+$	$-$	$-$	$-$	$+$	three variations;
for $r = +\infty$	$+$	$+$	$+$	$-$	$+$	$+$	two variations.

Again the equation has one real root.

The following tables give the values of r corresponding to different values of s, negative values of s being given by the first table, and positive values by the second.

TABLE 1 s negative		TABLE 2 s positive					
s	r	s	r	s	r	s	r
$-$		$+$		$+$		$+$	
0	0.6105	0					
.1	.6200	.1	0.6023	1.1	0.5577	2.1	0.5397
.2	.6306	.2	.5952	1.2	.5552	2.2	.5385
.3	.6431	.3	.5890	1.3	.5529	2.3	.5374
.4	.6577	.4	.5834	1.4	.5508	2.4	.5363
.5	.6744	.5	.5785	1.5	.5489	2.5	.5353
.6	.6936	.6	.5741	1.6	.5471	2.6	.5343
.7	.7150	.7	.5701	1.7	.5454	2.7	.5334
.8	.7364	.8	.5665	1.8	.5438	2.8	.5326
.9	.7523	.9	.5633	1.9	.5424	2.9	.5318
1.0	.5000	1.0	.5604	2.0	.5410	3.0	.5310

The following example will illustrate the application of these tables: Let the relative length, l, of a tube be 0.925; it is required to find its velocity when the values of a, b, and c are the same as those found in the Mississippi River experiments, viz:

$$a = +3.1950, \qquad b = +0.4424, \qquad c = -0.7652.$$

We have

$$s = \frac{b}{cl} = -\frac{0.4424}{0.7652 \times 0.925} = -0.625.$$

With the argument $s = -0.625$, we find from table 1, by interpolation, $r = 0.6989$, whence

$$h = rl = 0.6989 \times 0.925 = 0.6465;$$

and the velocity of the tube,

$$v' = a + bh + ch^2 = 3.161.$$

Assuming that the impulse and resistance of fluids varies as the first power of the velocity, the problem would be solved as follows: Since in this case the velocity of the tube would be the same as the mean velocity of the current taken in a vertical line from the surface to the depth 0.925, we should have the following expression for the velocity of the tube:

$$v' = \frac{1}{0.925}\int_0^{0.925}(a + bx + cx^2)\,dx = 3.181.$$

ARTICLE 3. Equations 1 and 2 and the preceding tables apply when the

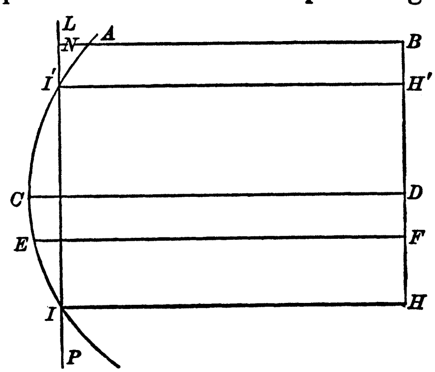

FIG. 3.

surface velocity of the water is greater than the tube velocity; but when the surface velocity is less than the tube velocity, a different equation is required

to express the conditions of equilibrium. Figure 3 gives a graphical representation of this case, in which the velocities are represented, as before, by horizontal lines drawn from BH to the parabolic curve. The surface velocity is also represented by AB, and the tube velocity by each of the lines IH and $I'H'$. EF denotes the velocity, v, of the water at any depth $BF = x$. As the axis of the tube, LP, intersects the velocity curve in the two points I and I', it is evident that the velocity of the water at each of these points is the same as the velocity of the tube. Putting $NI = h$ and $NI' = h_1$, we have the following two expressions for the velocity of the tube:

$$v' = a + bh + ch^2 = a + bh_1 + ch_1^2.$$

Since from N to I' and from I to P the velocity of the water is less than the velocity of the tube, while from I' to I the velocity of the water is greater than the tube velocity, we have, according to the principles explained in Article 3, the following equation of condition:

$$\int_0^{h_1} (v' - v)^2 dx + \int_h^l (v' - v)^2 dx = \int_{h_1}^h (v - v')^2 dx;$$

substituting for v and v' their values, integrating and arranging the results with reference to the powers of h, and remembering that $h + h_1 = -b/c$,* we have

$$\text{(3)} \quad \begin{aligned} &\frac{32}{15} h^5 + \left(\frac{32}{6}\frac{b}{c} - l\right) h^4 + \left(\frac{16}{3}\frac{b^2}{c^2} - 2\frac{b}{c} l\right) h^3 + \left(\frac{8}{3}\frac{b^3}{c^3} - \frac{b^2}{c^2} l + \frac{b}{c} l^2 + \frac{2}{3} l^3\right) h^2 \\ &\quad + \left(\frac{2}{3}\frac{b^4}{c^4} + \frac{b^2}{c^2} l^2 + \frac{2}{3}\frac{b}{c} l^3\right) h + \frac{b^5}{15c^5} - \frac{b^2}{c^2}\frac{l^3}{3} - \frac{b}{c}\frac{l^4}{2} - \frac{l^5}{5} = 0. \end{aligned}$$

Writing s for b/cl, and r for h/l, and dividing through by l^5, we obtain

$$\text{(4)} \quad \begin{aligned} &\frac{32}{15} r^5 + \left(\frac{32}{6} s - 1\right) r^4 + \left(\frac{16}{3} s^2 - 2s\right) r^3 + \left(\frac{8}{3} s^3 - s^2 + s + \frac{2}{3}\right) r^2 \\ &\quad + \left(\frac{2}{3} s^4 + s^2 + \frac{2}{3} s\right) r + \frac{s^5}{15} - \frac{s^2}{3} - \frac{s}{2} - \frac{1}{5} = 0. \end{aligned}$$

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* In order to prove this, we find, by differentiating the equation $v = a + bx + cx^2$, that the depth of maximum velocity is $x_1 = -\dfrac{b}{2c} = BD$ in figure 3, or $2\,BD = -\dfrac{b}{c} = BH + BH' = h + h_1$.

This equation will give only one real value of r corresponding to any real value of s; proceeding as before we find by Sturm's Theorem when $s = -0.75$,

	X	X_1	R	R_1	R_2	R_3	
	r^5	r^4	r^3	$-r^2$	r	$+1.36$	
for $r = -\infty$	$-$	$+$	$-$	$-$	$-$	$+$	three variations ;
for $r = +\infty$	$+$	$+$	$+$	$-$	$+$	$+$	two variations.

The equation has one real root.

The following table gives the values of r corresponding to different values of s:

TABLE 3 s negative

$-s$	r	$-s$	r	$-s$	r	$-s$	r	$-s$	r	$-s$	r
0.70	0.7149	0.80	0.7372	0.90	0.7645	1.00	0.8053	1.10	0.8644	1.20	0.9372
.71	.7171	.81	.7395	.91	.7679	1.01	.8104	1.11	.8713	1.21	.9449
.72	.7193	.82	.7419	.92	.7714	1.02	.8157	1.12	.8783	1.22	.9526
.73	.7215	.83	.7444	.93	.7750	1.03	.8212	1.13	.8854	1.23	.9603
.74	.7237	.84	.7470	.94	.7787	1.04	.8268	1.14	.8925	1.24	.9681
.75	.7259	.85	.7497	.95	.7826	1.05	.8326	1.15	.8997	1.25	.9759
.76	.7281	.86	.7524	.96	.7867	1.06	.8385	1.16	.9070	1.26	.9837
.77	.7303	.87	.7552	.97	.7910	1.07	.8446	1.17	.9144	1.27	.9915
.78	.7326	.88	.7582	.98	.7955	1.08	.8510	1.18	.9219	1.28	.9993
.79	.7349	.89	.7613	.99	.8003	1.09	.8576	1.19	.9295	1.29	1.0071

The velocity of a tube of given length, when $-b/c$ is greater than 2/3, can be found by the use of table 3, in the same manner as already shown in connection with tables 1 and 2.

ARTICLE 4. The problem of finding the velocity of a tube of given length is of less practical importance than that of finding the length of tube whose velocity shall be the same as the true mean velocity of the current in the same vertical line, and taken from the surface to the bottom of the stream.

We will call this the equivalent length, and designate it by L. Resuming equation (1), and arranging the terms with reference to the powers of l, and writing L for l, we have

$$(5)\quad \frac{L^5}{5} + \frac{b}{2c} L^4 + \left(\frac{b^2}{3c^2} - \frac{2}{3}\frac{e}{c}\right) L^3 - \frac{e}{c}\frac{b}{c} L^2 + \frac{e^2}{c^2} L - \frac{16}{15} h^5 - \frac{5}{3}\frac{b}{c} h^4 - \frac{2}{3}\frac{b^2}{c^2} h^3 = 0,$$

in which $e/c = (b/c)\, h + h^2$. In order to show that this equation has but one real root, substitute in it $-\,1/2$ for b/c, and we have

	X	X_1	R	R_1	R_2	R_3	
	L^5	L^4	L^3	$-L^2$	$-L$	$+0.06$	
for $L = -\infty$	$-$	$+$	$-$	$-$	$+$	$+$	three variations;
for $L = +\infty$	$+$	$+$	$+$	$-$	$-$	$+$	two variations.

The equation has one real root. In order that L in this equation may represent the equivalent length, h must be determined by the condition that the velocity of the water at the depth h shall be the same as the mean velocity of the current, taken in the same vertical line as the axis of the tube, and from the surface to the bottom of the stream. But this mean velocity is given by the following expression:

$$\int_0^1 v dx = \int_0^1 (a + bx + cx^2) dx = a + \frac{b}{2} + \frac{c}{3}.$$

Therefore the condition for determining h is

$$a + bh + ch^2 = a + \frac{b}{2} + \frac{c}{3}$$

or

$$h = -\frac{b}{2c} + \sqrt{\left(\frac{b}{2c}\right)^2 + \frac{b}{2c} + \frac{1}{3}}.$$

Using this value of h in equation (5) we obtain the following values of the roots, or equivalent lengths, corresponding to different values of b/c, and corresponding to the case discussed in Article 3.

TABLE 4

$\frac{b}{c}$	L	$\frac{b}{c}$	L	$\frac{b}{c}$	L	$\frac{b}{c}$	L
−		+		+		+	
0	0.946			1.0	0.967	2.0	0.976
.1	.942	0.1	0.949	1.1	0.968	2.1	0.977
.2	.939	.2	.951	1.2	.969	2.2	.977
.3	.935	.3	.954	1.3	.970	2.3	.978
.4	.930	.4	.956	1.4	.971	2.4	.978
.5	.927	.5	.958	1.5	.972	2.5	.979
.6	.925	.6	.960	1.6	.973	2.6	.979
		.7	.962	1.7	.974	2.7	.980
		.8	.964	1.8	.975	2.8	.980
		.9	.966	1.9	.976	2.9	.981

ARTICLE 5. We will next determine L when the velocity of the tube is greater than the surface velocity, which is the case discussed in article 3. The condition in this case, expressed analytically, is as follows:

$$a + bh + ch^2 > a, \qquad \text{or} \qquad -\frac{b}{c} > h,$$

and from the formula for h as given in article 4, we find that when $b/c = -2/3$, $h = 2/3$, and that when b/c is negative and numerically greater than $2/3$, $-b/c > h$. Therefore it is evident that for such values of b/c equation (3) must be used to determine L.

Arranging this equation with reference to the powers of l, and writing L for l, we have

$$\begin{aligned} &\frac{L^5}{5} + \frac{b}{2c}L^4 + \left(\frac{b^2}{3c^2} - \frac{2}{3}\frac{e}{c}\right)L^3 - \frac{e}{c}\frac{b}{c}L^2 + \frac{e^2}{c^2}L \\ (6) \quad &- \frac{2e^2}{c^2}\left(h - h_1\right) + \frac{2e}{c}\frac{b}{c}\left(h^2 - h_1^2\right) + \left(\frac{4e}{3c} - \frac{2}{3}\frac{b^2}{c^2}\right)\left(h^3 - h_1^3\right) \\ &- \frac{b}{c}\left(h^4 - h_1^4\right) - \frac{2}{5}\left(h^5 - h_1^5\right) = 0, \end{aligned}$$

in which $e/c = bh/c + h^2$, and $h_1 = -b/c - h$.

For any value of L given by equation (6), and corresponding to a given value of b/c and the following values of h and h_1, viz:

$$h = -\frac{b}{2c} + \sqrt{\left(\frac{b}{2c}\right)^2 + \frac{b}{2c} + \frac{1}{3}},$$

$$h_1 = -\frac{b}{2c} - \sqrt{\left(\frac{b}{2c}\right)^2 + \frac{b}{2c} + \frac{1}{3}},$$

there will also be a value of L given by equation (5), in which h_1 is to be used instead of h. It is seen by inspection that the coefficients of the corresponding powers of L in equations (5) and (6) are identical, for a given value of b/c.

Moreover these coefficients will be the same whether h or h_1 be used; for we have $h + h_1 = -b/c$, and multiplying both members by $h - h_1$, and transposing, we find

$$\frac{b}{c}h + h^2 = \frac{b}{c}h_1 + h_1^2 = \frac{e}{c}.$$

It follows that for the same values of b/c, equations (5) and (6) will differ only in their absolute terms. When $b/c = -2/3$, one value of L is zero; that is a surface float will give the true mean velocity.

Applying Sturm's Theorem to equation (6), after substituting -0.8 for b/c, we have

	X	X_1	R	R_1	R_2	R_3	
	L^5	L^4	L^3	L^2	$-L$	$+\ 0.45$	
for $L = -\infty$	$-$	$+$	$-$	$+$	$+$	$+$	three variations;
for $L = +\infty$	$+$	$+$	$+$	$+$	$-$	$+$	two variations.

The equation has one real root.

The following table gives the two values of L corresponding to the same value of b/c, the numbers in the second column being roots of equation (5), and those in the last column roots of equation (6):

TABLE 5

$\frac{-b}{c}$	L (5)	L (6)
0.7	0.050	0.929
.8	.196	.941
.9	.393	.951
1.0	.600	.929
1.01	.633	.918
1.02	.676	.900
1.03	.804	.804

ARTICLE 6. The method of determining the values of the constants, a, b, and c, in the formula $v = a + bx + cx^2$, will be briefly described. In the second column of the table at the end of article 1, are given the values of observed velocities at proportional depths, and from these data the equations of condition are written as follows:

$$\begin{array}{lcrcrcl}
a & & & & & = & 3.1950 \\
a & + & 0.1b & + & 0.01c & = & 3.2299 \\
a & + & .2b & + & .04c & = & 3.2532 \\
a & + & .3b & + & .09c & = & 3.2611 \\
a & + & .4b & + & .16c & = & 3.2516 \\
a & + & .5b & + & .25c & = & 3.2282 \\
a & + & .6b & + & .36c & = & 3.1807 \\
a & + & .7b & + & .49c & = & 3.1266 \\
a & + & .8b & + & .64c & = & 3.0594 \\
a & + & .9b & + & .81c & = & 2.9759
\end{array}$$

The normal equations, formed from the above equations of condition, are as follows:

$$\begin{array}{rcrcrcr}
10a & + & 4.5b & + & 2.85c & = & 31.7616 \\
4.5a & + & 2.85b & + & 2.025c & = & 14.0896 \\
2.85a & + & 2.025b & + & 1.5332c & = & 8.8288
\end{array}$$

The solution of these equations gives

$$a = +\,3.1952, \quad b = +\,0.4424, \quad c = -\,0.7652.$$

ARTICLE 7. It was stated in the first part of article 2 that the hypothesis of an impulse and resistance of fluids proportional to the first power of the velocity necessarily involves the assumption that the equivalent length of a tube is always equal to the total depth of the stream at the point of immersion.

As this case is very simple, we will give the analytical proof. The equation expressing the conditions of the equilibrium of the forces acting on the tube is

$$\int_0^h (v - v')\,dx = \int_h^l (v' - v)\,dx.$$

Substituting the values of v and v', integrating and reducing, we have

$$\frac{l^2}{3} + \frac{b}{2c}\,l = h^2 + \frac{b}{c}\,h.$$

In order that l, in this equation, may represent the equivalent length, we must have

$$h = -\frac{b}{2c} + \sqrt{\left(\frac{b}{2c}\right)^2 + \frac{b}{2c} + \frac{1}{3}}.$$

Substituting this value of h in the above equation, and writing L for l, we have

$$\frac{L^2}{3} + \frac{b}{2c}\,L = \frac{b}{2c} + \frac{1}{3},$$

whence $L = 1$. In the same manner it may be shown that the value of L derived from the equation,

$$\int_0^{h_1} (v' - v)\,dx + \int_h^l (v' - v)\,dx = \int_{h_1}^h (v - v')\,dx$$

is always equal to unity.

The value of the second root in this case is $L = -\left(1 + \dfrac{3}{2}\dfrac{b}{c}\right)$, and this is always positive and real and ranges from 0 to $\frac{1}{2}$.

When $b/c = -\,2/3$, its value is 0, which is the same value that obtains on the hypothesis of a resistance proportional to the square of the velocity.

ARTICLE 8. We have hitherto assumed that the tube in floating maintains a vertical position, and it is now necessary to prove that this is true within practical limits. For this purpose, we will develop a formula for finding the inclination of the tube. We shall consider only the case discussed in article 3, in which from the surface to the depth h the velocity of the water is greater than the velocity of the tube, while for depths exceeding h the velocity of the water is less than the tubular velocity.

Since in this case the moments of the forces acting on the tube all tend to produce rotation in the same direction, it will be the case in which the inclination of the tube is greatest. The case is represented in figure 1, the tube being denoted by LP; LN is the portion of the tube that projects out of the water, OP the weighted portion of the tube, and R its center of gravity; NP, the immersed portion of the tube, is the part that we have designated by l, or L; NI is h.

It is evident that from N to I the forces acting on the tube, at right angles to its axis, all act in the same direction. Call the sum of the moments of these forces taken with reference to I, M_1, and the sum of the moments of the forces acting in the opposite direction, from I to P, M_2. The sum of these two moments must be equal to the moment of stability of the tube, or $M_1 + M_2 = WC \sin f$, in which W is the weight of the tube, and is equal and opposite to the upward thrust of the displaced water, $C \sin f$ is the arm of the couple of these two forces, and f is the angle of deviation of the tube from the vertical. Whence

$$\sin f = \frac{M_1 + M_2}{WC}.$$

Prof. Rankine in his "Applied Mechanics," article 652, gives the following formula for the pressure of a current upon a solid body floating or immersed in it:

$$R = kD \frac{v^2}{2g} A,$$

in which R is the pressure in pounds, k a quantity depending on the figure of the body (equal for a cylinder moving sideways to about 0.77), D the weight of an unit of volume of the fluid (62.3 lbs. for water), v the velocity of the current in feet per second, g the acceleration of gravity, and A the greatest cross-section of the immersed portion of the body, taken at right angles to the direction of motion.

Substituting these values, the formula becomes

$$R = \frac{0.77 \times 62.3}{64.4} Av^2 = 0.745\, Av^2 .$$

In order to apply this to the present case, we write $(v - v')^2$ in place of v^2; calling r the radius of the tube, we have $A = \int 2rr'dx$, in which r' is the total depth in feet.

We may now write the following expression for the sum of the forces acting on the tube from N to I:

$$P_1 = 0.745 \int_0^h (v - v')^2\, 2rr'dx .$$

The sum of the moments of these forces, taken with reference to I, is

$$M_1 = 0.745 \times 2rr'^2 \int_0^h (v - v')^2 (h - x)\, dx.$$

We have also the following expression for the sum of the moments of the forces acting on IP, and taken with reference to the same center of moments, I:

$$M_2 = 0.745 \times 2rr'^2 \int_h^l (v' - v)^2 (x - h)\, dx.$$

Performing the indicated operations, and adding the expressions for M_1 and M_2, we have

$$\begin{aligned} M_1 + M_2 = 1.49\, rr'^2 c^2 \Big[&-\left(\frac{b^2}{c^2} h^3 + 2\frac{b}{c} h^4 + h^5\right) l + \left(\frac{3}{2}\frac{b^2}{c^2} h^2 + 2\frac{b}{c} h^3 + \frac{h^4}{2}\right) l^2 \\ &-\left(\frac{b^2}{c^2} h - \frac{2}{3} h^3\right) l^3 - \left(\frac{b}{c} h + \frac{h^2}{2} - \frac{b^2}{4c^2}\right) l^4 - \left(\frac{h}{5} - \frac{2}{5}\frac{b}{c}\right) l^5 \\ &+ \frac{b^2}{2c^2} h^4 + \frac{6}{5}\frac{b}{c} h^5 + \frac{11}{15} h^6 + \frac{l^6}{6} \Big]. \end{aligned}$$

Calling the quantity within the brackets B, we may write

$$M_1 + M_2 = 1.49 rr'^2 c^2 B.$$

If l is the equivalent length, B will be a function of b/c, and the following table gives some of its numerical values:

TABLE 6

$-b/c$	B
0	0.02328
.1	.01836
.2	.01410
.3	.01046
.4	.00736
.5	.00498
.6	.00318
$\frac{2}{3}$	.00231

Since the weight, W, of the tube is the same as the weight of the displaced water, $W = 62.3\pi r^2 r' l$.

The arm $C \sin f$, of the couple, which forms the moment of stability, is the horizontal distance between two vertical lines, one of which is the line drawn through the center of gravity, R, of the tube, and the other the line drawn through the center of gravity of the displaced fluid. Hence $C = (l/2 - RP)r'$, and putting $RP = l/m$, we have $C = (m-2)lr'/2m$. Substituting these values of $M_1 + M_2$, W, and C in the formula $\sin f = (M_1 + M_2)/WC$, we obtain

$$\sin f = \frac{2.98mc^2B}{62.3\pi rl^2(m-2)} = \frac{0.01523mc^2B}{rl^2(m-2)}.$$

Assuming $m = 5$ and $r = 1/12$, which were the proportions adopted in the Lowell Hydraulic Experiments, we have

$$\sin f = \frac{0.3045Bc^2}{L^2}.$$

By means of this formula was computed the value of f given in the table below. It is seen that the deviation from the vertical is very small, and that the tube may be regarded as practically in a vertical position.

The following table gives the results of the application of the preceding principles in determining v, L, and f. The data for the construction of the table were taken from "The Roorkee Hydraulic Experiments" of Capt. Allan

Cunningham, a reference to which was made in article 1. In the work describing these experiments Capt. Cunningham has given an able investigation of the theory of rod-motion, found on pages 240-246. The first column of the following table gives the number of the series of experiments, each being the mean of several trials. The second, third, and fourth columns, give the values of a, b, and c, respectively, as determined by the method of least squares.

The sixth and seventh columns give the values of the equivalent length L, which sometimes has one and sometimes two values, corresponding to the same value of b/c. The eighth column gives the value of f to the nearest minute, which is the deviation of the tube from the perpendicular.

ROORKEE HYDRAULIC EXPERIMENTS.

TABLE 7.

No. of Series	a	b	c	b/c	L		f	No. of Series	a	b	c	b/c	L		f
		+	—	—						+	—	—			
1	4.25	0.07	0.88	0.08	0.943		18′	21	4.44	0.03	0.57	0.05	0.944		8′
2	4.33	.33	1.15	.29	.934		17	22	3.51	.71	1.23	.58	.926		7
3	3.84	.36	1.25	.29	.935		20	23	4.29	.10	.83	.12	.942		14
4	3.48	.49	1.27	.39	.930		15	24	3.41	.26	.89	.29	.935		10
5	4.61	.72	1.11	.65	.925	0	4	25	3.89	.34	.96	.35	.932		10
6	4.27	.65	.99	.66	.925	0	3	26	2.94	.67	1.39	.48	.927		13
7	4.07	.52	.99	.53	.926		5	27	3.34	.30	.85	.35	.932		8
										—		+			
8	4.06	.38	.89	.43	.929		6	28	2.84	.03	.60	.05	.947		
										+		—			
9	4.31	.49	.92	.53	.926		5	29	2.39	1.12	1.01	1.11	No equiv. length in this case.		
10	4.50	.16	.55	.29	.935		4	30	2.46	1.49	1.50	.99	.931	0.579	
11	4.05	.51	.97	.53	.926		5	31	2.74	1.09	1.12	.97	.936	.538	
12	3.81	.47	1.20	.39	.930		13	32	3.07	1.43	1.57	.91	.949	.413	
13	4.06	.55	1.04	.53	.926		6	33	3.15	1.34	1.32	1.02	.900	.676	
14	3.89	.84	1.26	.67	.925	0	4	34	3.55	1.64	2.09	.78	.939	.167	
15	4.10	.47	.92	.51	.926		5	35	4.18	.73	1.36	.54	.926		10
16	3.99	.47	1.07	.44	.929		9	36	4.16	.61	1.28	.48	.928		11
17	3.76	.56	1.12	.50	.927		8	37	4.12	.26	.93	.28	.936		12
18	6.43	.43	.72	.60	.926		2	38	3.63	1.24	1.58	.78	.939	.167	
		—		+											
19	6.05	.33	.11	3.00	.981			39	3.91	.65	1.14	.57	.926		6
		+		—											
20	5.65	.16	.85	.20	.939		12	40	3.15	1.35	1.72	.78	.939	.167	

BOSTON, MASS.,
JULY, 1909.